The Wald entropy and 6d conformal anomaly

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Abstract

We analyze the Wald entropy for different forms of the conformal anomaly in six dimensions. In particular we focus on the anomaly which arises in a holographic calculation of Henningson and Skenderis. The various presentations of the anomaly differ by some total derivative terms. We calculate the corresponding Wald entropy for surfaces which do not have an Abelian $O(2)$ symmetry in the transverse direction although the extrinsic curvature vanishes. We demonstrate that for this class of surfaces the Wald entropy is different for different forms of the conformal anomaly. The difference is due to the total derivative terms present in the anomaly. We analyze the conformal invariance of the Wald entropy for the holographic conformal anomaly and demonstrate that the violation of the invariance is due to the contributions of the total derivative terms in the anomaly. Finally, we make more precise the general form for the Hung-Myers-Smolkin discrepancy.

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1 Introduction

More than twenty years ago Wald has introduced a very efficient prescription by which the gravitational entropy can be computed for a given gravitational action [1], [2] (see also [3]). The typical gravitational action in question is polynomial in the Riemann curvature and its covariant derivatives. This prescription associates the entropy with a horizon, a co-dimension two surface with very peculiar properties. In particular, it is assumed that there exists a time-like Killing vector which becomes null on this surface. This vector generates an Abelian symmetry in the transverse direction to the surface. As a consequence, the extrinsic curvature of the surface is vanishing.

In a related concept of the entanglement entropy one associates some entropy with arbitrary co-dimension two surface, not necessarily a horizon. The presence of the Abelian symmetry is thus not guaranteed in this case. Nevertheless, there appears to be a certain algorithm [7], [8], [9], [10] to associate some entropy with each individual term, dependent on the curvature, in the quantum effective action. This algorithm certainly deviates from the one proposed by Wald if the surface in question is characterized by a non-trivial extrinsic curvature. However, what happens if the extrinsic curvature is vanishing while the Abelian symmetry is not present? This would be the situation when the Wald algorithm for the entropy might be thought to still work. This was tested by Hung, Myers and Smolkin (HMS) in [4] for a certain class of six-dimensional geometries with the properties as above where they compared the Wald entropy due to the $6d$ conformal anomaly with the holographic entropy computed by using the holographic prescription by Ruy and Takanayagi [6]. They have found a discrepancy between these two calculations. In the recent works [5] we have made some attempts to explain this discrepancy, for the alternative attempts see [14].

In the present note we deliberately ignore the problem of finding an explanation for the HMS discrepancy. Instead, we make a step back and ask the question which should have been answered first: what is the general form for the discrepancy? The answer to this question is not that trivial as it would seem to be. The reason is the following. In [4] it was considered not the most general form for the conformal anomaly and certainly not the one which actually appears in a holographic calculation as, for instance, in the paper of Henningson and Skenderis [11]. The difference between the two possible forms is due to the total derivatives. In this note we show that what the Wald entropy is concerned these total derivatives are important and can not be disregarded.

This note is organized as follows. In section 2 we present the general form for the trace anomaly in six dimensions. This general form includes the Euler number, three conformal invariants and a set of total derivative terms. In section 2.2 the holographic conformal anomaly calculated by Henningson and Skenderis is presented in this general form and we give the values of the respective conformal charges as well as the charges that correspond to the total

derivatives. In section 3 we give the formulas for the Wald entropy due to the 6d conformal anomaly, both in a general form and for the holographic conformal anomaly. In section 4 we consider the HMS geometries and surfaces and compute the Wald entropy for the holographic anomaly presented in two forms, the one in which it appears in the holographic calculation and the other which involves the representation in terms of the conformal invariants, and find a mismatch. In section 5 we explain the mismatch by taking into account the contribution of the total derivatives. The Wald entropy of the latter is non-vanishing in general. In section 6 we discuss the conformal invariance of the corresponding entropy. In section 7 we give a general form for the HMS discrepancy. We conclude in section 8.

2 Conformal anomaly in six dimensions

2.1 General form for the anomaly

In a generic conformal field theory in $d=6$ the trace anomaly, modulo the total derivatives, is a combination of four different terms

$$\begin{aligned} &< T >= \mathcal{A} = \mathcal{A}_{BI} + \mathcal{A}_C \, , \\ &\mathcal{A}_{BI} = aE_6 + b_1 I_1 + b_2 I_2 + b_3 I_3 \, , \quad \mathcal{A}_C = \sum_{k=1}^{7} d_k C_k \, , \end{aligned} \tag{1}$$

where E_6 is the Euler density in $d=6$ and, using notations of [12], we have

$$\begin{aligned} &I_1 = W_{\alpha\mu\nu\beta} W^{\mu\sigma\rho\nu} W_{\sigma\ \ \rho}^{\ \alpha\beta} \, , \\ &I_2 = W_{\alpha\beta}^{\ \ \ \mu\nu} W_{\mu\nu}^{\ \ \ \sigma\rho} W_{\sigma\rho}^{\ \ \ \alpha\beta} \, , \\ &I_3 = W_{\mu\alpha\beta\gamma} \Box W^{\mu\alpha\beta\gamma} + W_{\mu\alpha\beta\gamma} (4R^\mu_\nu - \frac{6}{5} R\delta^\mu_\nu) W^{\nu\alpha\beta\gamma} \, , \end{aligned} \tag{2}$$

where $W_{\alpha\beta\mu\nu}$ is the Weyl tensor. In (1) the part $\mathcal{A}_C$ is due to the total derivatives

$$\begin{aligned} &C_1 = B_1 \, , C_2 = B_2 + B_6 \, , \, C_3 = B_3 + B_7 \, , \\ &C_4 = B_4 + B_8 \, , \, C_5 = B_5 + B_9 \, , \\ &C_6 = \frac{1}{9} B_2 - B_4 - \frac{1}{5} B_{11} - \frac{3}{2} B_{13} + B_{14} \, , \\ &C_7 = \frac{1}{60} B_2 - \frac{3}{4} B_3 + \frac{3}{4} B_4 + \frac{1}{4} B_5 + \frac{1}{12} B_{12} + \frac{1}{2} B_{15} - \frac{1}{4} B_{16} - B_{17} \, , \end{aligned} \tag{3}$$

where

$$\begin{aligned} &B_1 = \nabla^4 R \, , \, B_2 = (\nabla_\alpha R)^2 \, , \, B_3 = (\nabla_\alpha B_{\mu\nu})^2 \, , \, B_4 = \nabla_\alpha B_{\mu\nu} \nabla^\mu B^{\alpha\nu} \, , \, B_5 = (\nabla_\alpha W_{\mu\nu\rho\sigma})^2 \, , \\ &B_6 = R \nabla^2 R \, , \, B_7 = B_{ab} \nabla^2 B^{ab} \, , \, B_8 = B_{\alpha\beta} \nabla_\mu \nabla^\beta B^{\alpha\mu} \, , \, B_9 = W_{\alpha\beta\mu\nu} \nabla^2 W^{\alpha\beta\mu\nu} \, , \, B_{10} = R^3 \, , \\ &B_{11} = R B^2_{\mu\nu} \, , \, B_{12} = R W^2_{\mu\nu\rho\sigma} \, , \, B_{13} = B_\mu^{\ \nu} B_\nu^{\ \rho} B_\rho^{\ \mu} \, , \, B_{14} = B_{\alpha\beta} B_{\mu\nu} W^{\alpha\mu\beta\nu} \\ &B_{15} = B_{\alpha\beta} W^{\alpha\mu\nu\rho} W^\beta_{\ \ \mu\nu\rho} \, , \, B_{16} = W_{\alpha\beta}^{\ \ \ \mu\nu} W_{\mu\nu}^{\ \ \ \rho\sigma} W_{\rho\sigma}^{\ \ \ \alpha\beta} \, , \, B_{17} = W_{\alpha\mu\beta\nu} W^{\alpha\rho\beta\sigma} W^\mu_{\ \ \rho\ \ \sigma}^{\ \ \nu} \, , \end{aligned} \tag{4}$$

and the tensor B is defined as

$$B_{\mu\nu} = R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu} \,. \tag{5}$$

The form (1), when the contribution of the total derivative terms C_k is neglected, for the conformal anomaly we shall call the BI-form.

2.2 Holographic conformal anomaly

In this paper, we are mostly interested in the holographic conformal anomaly computed in [11]. It is derived by a standard holographic procedure from the 7-dimensional gravitational action

$$W_7 = -\frac{1}{2\ell_p^5} \int d^7x \sqrt{g} (R + \frac{30}{L^2}) \,. \tag{6}$$

This anomaly takes the form

$$\mathcal{A}^h = -\frac{L^5}{64\ell_p^5} \big(\frac{1}{2} R R_{\mu\nu} R^{\mu\nu} - \frac{3}{50} R^3 - R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + \frac{1}{5} R^{\mu\nu} \nabla_\mu \nabla_\nu R - \frac{1}{2} R^{\mu\nu} \Box R_{\mu\nu} + \frac{1}{20} R \Box R \big) \,. \tag{7}$$

It can be represented in the form (1) as follows

$$\begin{aligned}
\mathcal{A}^h &= \mathcal{A}^h_{BI} + \mathcal{A}^h_C \,, \\
\mathcal{A}^h_{BI} &= \frac{3L^5}{2^5\, 7!\ell_p^5} (-\frac{35}{2} E_6 - 1680 I_1 - 420 I_2 + 140 I_2) \,, \\
\mathcal{A}^h_C &= \frac{3L^5}{2^5\, 7!\ell_p^5} (-140 C_5 + 420 C_3 - 504 C_4 - 84 C_6 + 560 C_7) \,.
\end{aligned} \tag{8}$$

So that we find for the charges b_k and a,

$$b_1 = -\frac{L^5}{32\ell_p^5} \,, \ b_2 = -\frac{L^5}{128\ell_p^5} \,, \ b_3 = \frac{L^5}{384\ell_p^5} \,, \ a = -\frac{L^5}{3072\ell_p^5} \tag{9}$$

in agreement with [4] and [13]. We also find that

$$d_3 = -\frac{L^5}{384\ell_p^5} \,, \ d_4 = -\frac{3L^5}{320\ell_p^5} \,, \ d_5 = -\frac{L^5}{384\ell_p^5} \,, \ d_6 = -\frac{L^5}{640\ell_p^5} \,, \ d_7 = \frac{L^5}{96\ell_p^5} \tag{10}$$

and all other d_k vanish.

3 The Wald entropy

3.1 General formula

Suppose that the gravitational action W is a function of the Riemann curvature and its covariant derivatives (and contains maximum two derivatives). Then the corresponding Wald entropy is

computed according to the formula [2]

$$S_W = -2\pi \int_\Sigma (\frac{\partial W}{\partial R_{\mu\nu\rho\sigma}} - \nabla_\alpha \frac{\partial W}{\partial \nabla_\alpha R_{\mu\nu\rho\sigma}} + \nabla_{(\alpha}\nabla_{\beta)} \frac{\partial W}{\partial \nabla_{(\alpha}\nabla_{\beta)} R_{\mu\nu\rho\sigma}})\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{11}$$

where $\epsilon_{\mu\nu} = n^a_\mu n^b_\nu \epsilon_{ab}\,,\ n^a_\mu\,,\ a = 1, 2$ is a pair vectors normal to surface Σ, so that we have that

$$\epsilon_{\mu\nu}\epsilon_{\rho\sigma} = (n^a_\mu n^a_\rho)(n^b_\nu n^b_\sigma) - (n^a_\mu n^a_\sigma)(n^b_\nu n^b_\rho)\,. \tag{12}$$

We notice the symmetrization in the last term in (11). It will be important when we discuss the Wald entropy due to the total derivatives.

3.2 Entropy due to the holographic conformal anomaly

Using the above definition we compute the corresponding Wald's entropy. In the case of the holographic conformal anomaly (7) we find

$$\begin{aligned} S^h_W = {}& \frac{\pi L^5}{32\ell_p^5} \int_\Sigma [(RR_{aa} + R^2_{\mu\nu}) - \frac{9}{25}R^2 - (2R_{\mu a\nu a}R^{\mu\nu} + R^2_{aa} - R^2_{ab}) \\ & + \frac{1}{5}\nabla_a\nabla_a R - \Box R_{aa} + \frac{2}{5}\Box R]\,, \end{aligned} \tag{13}$$

where the indexes a and b correspond to projections on the normal vectors n^μ_a, $a = 1, 2$ and we introduced the notations $\nabla_a \equiv n^\mu_a \nabla_\mu\,,\ \Box R_{aa} \equiv n^\mu_a n^\nu_a \Box R_{\mu\nu}\,.$

3.3 Entropy due to conformal anomaly in the BI-form

On the other hand, for a generic anomaly in the form $\mathcal{A}_{BI}$ (1) we have that

$$S_{BI} = (aE_4 + b_1 s_1 + b_2 s_2 + b_3 s_3)\,, \tag{14}$$

$$\begin{aligned} s_1 &= -6\pi(W^{b\mu\nu a}W_\mu{}^{ab}{}_\nu - W^{a\mu\nu a}W_\mu{}^{bb}{}_\nu - \frac{1}{4}W^{a\mu\nu\sigma}W^a{}_{\mu\nu\sigma} + \frac{1}{20}W^{\mu\nu\sigma\rho}W_{\mu\nu\sigma\rho})\,, \\ s_2 &= -6\pi(2W^{ab\mu\nu}W_{\mu\nu}{}^{ab} - W^{a\mu\nu\sigma}W^a{}_{\mu\nu\sigma} + \frac{1}{5}W^{\mu\nu\sigma\rho}W_{\mu\nu\sigma\rho})\,, \\ s_3 &= -8\pi(\Box W^{abab} + 4R^a_\mu W^{\mu bab} - R_{\mu\nu}W^{\mu a\nu a} - \frac{6}{5}RW^{abab} + W^a{}_{\mu\nu\sigma}W^{a\mu\nu\sigma} - \frac{3}{5}W_{\mu\nu\sigma\rho}W^{\mu\nu\sigma\rho})\,, \end{aligned} \tag{15}$$

as was first derived in [4]. Notice that the contribution of the total derivative terms C_k is neglected in (14). This contribution is indeed supposed to vanish when considered in the standard situation of a Killing horizon.

4 The Wald entropy for the HMS-surfaces

4.1 The HMS geometries/surfaces

Hung, Myers and Smolkin [4] have considered the following six-dimensional geometries and the four-dimensional surfaces:

- **a)** $\Sigma : S^1 \times S^3$ in $R \times S^2 \times S^3$, R_1 is radius of S^3, R_2 is radius of S^2 ,
- **a')** $\Sigma : S^2 \times S^2$ in $R \times S^2 \times S^3$, R_1 is radius of S^3, R_2 is radius of S^2 ,
- **b)** $\Sigma : R^2 \times S^2$ in $R^3 \times S^3$, R_1 is radius of S^3 ,
- **c)** $\Sigma : R^1 \times S^3$ in $R^2 \times S^4$, R_1 is radius of S^4 .

All these geometries are characterized by same properties:

i) they are products of constant curvature spaces; that is why, in the Wald entropy (13) or (15) all terms with covariant derivatives vanish;

ii) the time coordinate $x^1 = t$ lies in the flat sub-space, so that components of the Riemann tensor $R_{\mu\nu\alpha\beta}$ or the Ricci tensor $R_{\mu\nu}$ vanish if one of the indexes is 1;

iii) the surface Σ has two normal vectors, one of which, n^1, is time-like. The corresponding extrinsic curvature vanishes because it is the Killing vector. The surface Σ has a component which is the minimal surface embedded in a sphere. The corresponding normal vector, n^2, lies along this sphere. The corresponding extrinsic curvature vanishes since it is the minimal surface;

vi) in particular, we have that $R^2_{aa} = R^2_{ab}$ and $R_{abcb} = 0$ for all these geometries;

v) in all these cases the Abelian $O(2)$ symmetry in the transverse direction to Σ is absent although the components of the extrinsic curvature of Σ vanish.

4.2 Entropy due to holographic conformal anomaly

First, we compute the entropy (13) due to the holographic conformal anomaly:

$$\mathbf{a}).\quad S^h_W = \frac{\pi}{400}\frac{l^5}{l_p^5}V_\Sigma\left(\frac{7}{R_2^4} - \frac{12}{R_1^4} - \frac{33}{R_1^2R_2^2}\right), \tag{16}$$

$$\mathbf{a'}).\quad S^h_W = \frac{\pi}{400}\frac{l^5}{l_p^5}V_\Sigma\left(\frac{7}{R_2^4} + \frac{38}{R_1^4} - \frac{58}{R_1^2R_2^2}\right), \tag{17}$$

$$\mathbf{b}).\quad S^h_W = \frac{19\pi}{200}\frac{l^5}{l_p^5}\frac{V_\Sigma}{R_1^4}, \tag{18}$$

$$\mathbf{c}).\quad S^h_W = \frac{27\pi}{400}\frac{l^5}{l_p^5}\frac{V_\Sigma}{R_1^4}, \tag{19}$$

where V_Σ is volume of Σ.

4.3 Entropy due to holographic conformal anomaly in BI-form

Now we compute the Wald entropy for the same holographic anomaly but represented in the BI form (8). We find

$$\mathbf{a}).\quad S^h_{BI} = \frac{\pi}{400}\frac{l^5}{l_p^5}V_\Sigma(\frac{7}{R_2^4} - \frac{12}{R_1^4} - \frac{33}{R_1^2R_2^2})\,, \tag{20}$$

$$\mathbf{a}').\quad S^h_{BI} = \frac{\pi}{400}\frac{l^5}{l_p^5}V_\Sigma(\frac{7}{R_2^4} + \frac{64}{3R_1^4} - \frac{58}{R_1^2R_2^2})\,, \tag{21}$$

$$\mathbf{b}).\quad S^h_{BI} = \frac{4\pi}{75}\frac{l^5}{l_p^5}\frac{V_\Sigma}{R_1^4}\,, \tag{22}$$

$$\mathbf{c}).\quad S^h_{BI} = -\frac{23\pi}{400}\frac{l^5}{l_p^5}\frac{V_\Sigma}{R_1^4}\,. \tag{23}$$

These results for the Wald entropy agree with the entropy computed in [4] when we choose the particular values (9) for the conformal charges b_k.

4.4 The mismatch

Clearly, we have a mismatch, $\Delta S^h = S^h_W - S^h_{BI}$, between these two calculations:

$$\mathbf{a}).\quad \Delta S^h = 0\,, \tag{24}$$

$$\mathbf{a}').\quad \Delta S^h = \frac{\pi}{24}\frac{l^5}{l_p^5}\frac{V_\Sigma}{R_1^4}\,, \tag{25}$$

$$\mathbf{b}).\quad \Delta S^h = \frac{\pi}{24}\frac{l^5}{l_p^5}\frac{V_\Sigma}{R_1^4}\,, \tag{26}$$

$$\mathbf{c}).\quad \Delta S^h = \frac{\pi}{8}\frac{l^5}{l_p^5}\frac{V_\Sigma}{R_1^4}\,. \tag{27}$$

5 Contribution of the total derivative terms

5.1 Some general formulas

Trying to understand the mismatch which we have just observed we have to consider carefully the total derivative terms which are in general present in the conformal anomaly. As wee see from (8) only terms $C_k\,,\ k = 3, 4, 5, 6, 7$ contribute to the holographic conformal anomaly. Before using the Wald formula (11) and compute the entropy it is required to first transform each C_k to a form which would contain terms symmetric in second covariant derivatives of the curvature. The terms C_3 and C_5 already take this form. The corresponding Wald entropy calculated using (11) gives zero entropy,

$$S^{C_3}_W = S^{C_5}_W = 0\,. \tag{28}$$

For the other terms the situation is less trivial. Commuting the covariant derivatives one finds

$$\begin{aligned}
C_4 &= D_4 + R_\alpha{}^\beta R_\beta{}^\mu R_\mu{}^\alpha - R^{\alpha\beta}R^{\mu\nu}R_{\alpha\mu\beta\nu} \\
C_7 &= D_7 + I_1 - \frac{1}{4}I_2 + \frac{1}{2}R_{\alpha\beta}W^{\alpha\mu\nu\rho}W^\beta{}_{\mu\nu\rho}\,, \\
C_6 &= D_6 + R_{\mu\nu}R_{\alpha\beta}R^{\mu\alpha\nu\beta} - R_\mu{}^\alpha R_\alpha{}^\beta R_\beta{}^\mu\,,
\end{aligned} \tag{29}$$

where I_1 and I_2 are defined in (12) and we introduced

$$\begin{aligned}
D_4 &= -\frac{1}{18}R\Box R - \frac{5}{36}(\nabla_\alpha R)^2 + \frac{1}{3}R^{\alpha\beta}\nabla_\alpha\nabla_\beta R + \nabla_\alpha R_{\mu\nu}\nabla^\nu R^{\alpha\mu}\,, \\
D_6 &= \frac{1}{4}(\nabla_\alpha R)^2 - \nabla_\mu R^{\alpha\beta}\nabla_\beta R^\mu{}_\alpha\,, \\
D_7 &= \frac{1}{4}(\nabla_\alpha R_{\mu\nu\rho\sigma})^2 - (\nabla_\alpha R_{\mu\nu})^2 + \frac{1}{16}(\nabla_\alpha R)^2 + \frac{3}{4}\nabla_\mu R^{\alpha\beta}\nabla_\beta R^\mu{}_\alpha\,.
\end{aligned} \tag{30}$$

Now we apply the general formula (11) and compute the Wald entropy,

$$\begin{aligned}
S_W^{(C_4)} &= 2\pi\int_\Sigma [-\frac{2}{3}(\Box - \nabla_a^2)R + (R^2_{aa} - R^2_{ab} - R^2_{\mu a})]\,, \qquad (31) \\
S_W^{(C_6)} &= 2\pi\int_\Sigma [(\Box - \nabla_a^2)R - (R^2_{aa} - R^2_{ab} - R^2_{\mu a})]\,, \\
S_W^{(C_7)} &= s_1 - \frac{1}{4}s_2 + 2\pi\int_\Sigma [\Box(R_{abab} - 2R_{aa} + \frac{1}{4}R) + \frac{3}{4}\nabla_a^2 R + \frac{3}{2}(R^2_{\mu a} - R_{\mu\nu}R^{\mu a\nu a}) \\
&-2\pi\int_\Sigma (\frac{1}{2}W^{a\mu\nu\rho}W^a{}_{\mu\nu\rho} + 2R^a_\mu W^{\mu bab} - \frac{1}{2}R_{\mu\nu}W^{\mu a\nu a})\,,
\end{aligned}$$

where we define $\Box R_{abab} = n^\mu_a n^\nu_a n^\alpha_b n^\beta_b \Box R_{\mu\alpha\nu\beta}$ and $\Box R_{aa} = n^\mu_a n^\nu_a \Box R_{\mu\nu}$, and the terms s_1 and s_2 have been defined in (15). We used the Bianchi identities and, in particular, that

$$\nabla_\alpha\nabla_\nu R^\alpha{}_\mu - \frac{1}{2}\nabla_\nu\nabla_\mu R = R_{\alpha\mu}R^\alpha{}_\nu - R^\alpha{}_{\mu\beta\nu}R^\beta{}_\alpha \tag{32}$$

when derived (31). This identity, in particular, shows that the right hand side of (32), and its projections on the transverse subspace, vanishes for a product of constant curvature spaces. We then use that if the extrinsic curvature of surface Σ vanishes we have the following relation

$$(\Box - \nabla_a^2)R = \Delta_\Sigma R\,, \tag{33}$$

where Δ_Σ is the Laplace operator defined on surface Σ and R_Σ is the intrinsic curvature of Σ. The integration of (33) over a closed surface Σ gives zero and the expressions (31) are simplified

$$\begin{aligned}
S_W^{(C_4)} &= -S_W^{(C_6)} = 2\pi\int_\Sigma (R^2_{aa} - R^2_{ab} - R^2_{\mu a})\,, \qquad (34) \\
S_W^{(C_7)} &= s_1 - \frac{1}{4}s_2 + \pi\int_\Sigma [2\Box(R_{abab} - 2R_{aa} + R) + 3(R^2_{\mu a} - R_{\mu\nu}R^{\mu a\nu a}) \\
&-W^{a\mu\nu\rho}W^a{}_{\mu\nu\rho} - 4R^a_\mu W^{\mu bab} + R_{\mu\nu}W^{\mu a\nu a}]\,.
\end{aligned}$$

We note that these formulas are valid for arbitrary geometry and any entangling surface for which the extrinsic curvature vanishes. If the geometry in question is a product of constant curvature spaces, as in the examples considered in section 4.1, then the terms with derivatives in (34) vanish. Each entropy in (34) is a priori non-vanishing and we shall demonstrate this below for the concrete examples. Accidentally, we observe that the Wald entropy for invariants C_4 and C_6 is the same, up to a sign.

5.2 Explaining the mismatch

Now let us compute the contribution of these to total derivative terms to the entropy, for various geometries discussed before. Our aim is to demonstrate that

$$\Delta S^h = (d_4 S_W^{(C_4)} + d_6 S_W^{(C_6)} + d_7 S_W^{(C_7)})\,. \tag{35}$$

The values for the charges d_k are given in (10).

We find in the following cases:

a) $\Sigma : S^1 \times S^3$ in $R \times S^2 \times S^3$:

$$S_W^{(C_4)} = -S_W^{(C_6)} = -\frac{2\pi V_\Sigma L^5}{\ell_p^5 R_2^4}\,, \quad S_W^{(C_7)} = -\frac{3\pi V_\Sigma L^5}{2\ell_p^5 R_2^4}\,, \tag{36}$$

and hence

$$\sum_k d_k S_W^{(C_k)} = 0\,. \tag{37}$$

a') $\Sigma : S^2 \times S^2$ in $R \times S^2 \times S^3$:

$$S_W^{(C_4)} = -S_W^{(C_6)} = -\frac{8\pi V_\Sigma L^5}{\ell_p^5 R_1^4}\,, \quad S_W^{(C_7)} = -\frac{2\pi V_\Sigma L^5}{\ell_p^5 R_1^4}\,, \tag{38}$$

and hence

$$\sum_k d_k S_W^{(C_k)} = \frac{\pi V_\Sigma L^5}{24\ell_p^5 R_1^4}\,. \tag{39}$$

b) $\Sigma : R^2 \times S^2$ in $R^3 \times S^3$:

$$S_W^{(C_4)} = -S_W^{(C_6)} = -\frac{8\pi V_\Sigma L^5}{\ell_p^5 R_1^4}\,, \quad S_W^{(C_7)} = -\frac{2\pi V_\Sigma L^5}{\ell_p^5 R_1^4}\,, \tag{40}$$

and hence

$$\sum_k d_k S_W^{(C_k)} = \frac{\pi V_\Sigma L^5}{24\ell_p^5 R_1^4} \, . \tag{41}$$

c) $\Sigma : R^1 \times S^3$ in $R^2 \times S^4$:

$$S_W^{(C_4)} = -S_W^{(C_6)} = -\frac{18\pi V_\Sigma L^5}{\ell_p^5 R_1^4} \, , \quad S_W^{(C_7)} = -\frac{3\pi V_\Sigma L^5}{2\ell_p^5 R_1^4} \, , \tag{42}$$

and hence

$$\sum_k d_k S_W^{(C_k)} = \frac{\pi V_\Sigma L^5}{8\ell_p^5 R_1^4} \, . \tag{43}$$

We see that in all these cases we obtain a complete agreement with (24)-(27) and confirm (35). The mismatch thus is indeed due to the contributions of the total derivative terms which appear in the conformal anomaly when we pass from the form (7) to the BI-form (8).

6 Conformal invariance

In this section we would like to check whether the Wald entropy in question is conformal invariant. We restrict ourselves to the infinitesimal conformal transformations, $\delta_\sigma g_{\mu\nu} = g_{\mu\nu}\sigma$. Then we find in six dimensions that

$$\begin{aligned}
&\delta_\sigma R = -5\Box\sigma - R\sigma \, , \ \delta_\sigma R_{\mu\nu} = -2\nabla_\mu\nabla_\nu\sigma - \frac{1}{2}g_{\mu\nu}\Box\sigma \, , \ \delta_\sigma R_{aa} = -2\nabla_a^2\sigma - \Box\sigma - \sigma R \, , \\
&\delta_\sigma R_{\mu a\nu a} = -\frac{1}{2}(g_{\mu\nu}\nabla_a^2\sigma - n_\nu^a\nabla_a\nabla_\mu\sigma - n_\mu^a\nabla_a\nabla_\nu\sigma + 2\nabla_\mu\nabla_\nu\sigma) \, .
\end{aligned} \tag{44}$$

First, we analyze the conformal invariance of the Wald entropy (13) due to the holographic conformal anomaly. In order to simplify the transformations we consider the case of a geometry with constant curvature, i.e. the condition $\nabla_\rho R_{\alpha\beta\mu\nu} = 0$ is assumed to be valid. Then we find for the variations of the quantities, integrated over the surface Σ,

$$\begin{aligned}
&\delta_\sigma \int_\Sigma \Box R = -\int_\Sigma (5\Box^2\sigma + R\Box\sigma) \, , \\
&\delta_\sigma \int_\Sigma \nabla_a^2 R = -\int_\Sigma (5\nabla_a^2\Box\sigma + R\nabla_a^2\sigma) \, , \\
&\delta_\sigma \int_\Sigma \Box R_{aa} = -\int_\Sigma (2\Box\nabla_a^2\sigma + \Box^2\sigma + R_{aa}\Box\sigma) \, , \\
&\delta_\sigma \int_\Sigma \Box R_{abab} = -\int_\Sigma (R_{abab}\Box\sigma + \Box\nabla_a^2\sigma) \, .
\end{aligned} \tag{45}$$

Since the extrinsic curvature of Σ vanishes we have $\Box\sigma = \nabla_a^2\sigma + \Delta_\Sigma\sigma$. Finally, we use the commutation relation

$$(\Box\nabla_a^2 - \nabla^2\Box)\sigma = 2(R_{ab} - R_{acbc})\nabla_a\nabla_b\sigma\,, \tag{46}$$

where $\nabla_a = n_a^\mu\nabla_\mu$, $a = 1, 2$ are derivatives in the transverse sub-space and we neglected all terms with derivatives along the surface Σ since, after integration over Σ, these terms give zero. Putting everything together we find

$$\delta_\sigma S_W^h = -\frac{\pi L^5}{8\ell_p^5}\int_\Sigma [(R_{ab} - \frac{1}{2}R_{cc}\delta_{ab})\nabla_a\nabla_b\sigma] \tag{47}$$

for the conformal transformation of the Wald entropy (13) due to the holographic conformal anomaly. This conformal transformation vanishes in many particular cases. For instance, it vanishes if the six-dimensional space-time is Einstein, i.e. $R_{\mu\nu} = \Lambda g_{\mu\nu}$. In the presence of $O(2)$ symmetry in the transverse sub-space we have that $R_{ab} = \lambda\delta_{ab}$, $a = 1, 2$ and the tensor $R_{ab} - \frac{1}{2}R_{cc}\delta_{ab} = 0$ so that (47) vanishes, as expected. However, for the geometries considered in section 4.1 we have that $R_{11} = 0$ and $R_{22} \neq 0$. In this case the right hand side of (47) is non-zero. It is proportional to $R_{22}(\nabla_2^2 - \nabla_1^2)\sigma$. This quantity is non-vanishing if σ is a generic function of coordinates x^1 and x^2 orthogonal to surface Σ.

The terms (15) in the Wald entropy that are due to the conformal invariants I_1, I_2 are conformal invariant by construction. Conformal invariance of entropy s_3 which is due to invariant I_3 is less obvious since s_3 is expressed not only in terms of the Weyl tensor but also in terms of the Ricci tensor, Ricci scalar and a Laplacian. On a background of a constant curvature geometry we find for the conformal variation of s_3,

$$\delta_\sigma s_3 = -24\pi\int_\Sigma(\Box\sigma W_{abab} - 2\nabla_a\nabla_b\sigma W_{acbc}) = -24\pi\int_\Sigma(\delta_{ab}\nabla_d^2\sigma - 2\nabla_a\nabla_b\sigma)R_{acbc}\,, \tag{48}$$

where in the second equality we have used that $\Box\sigma = \Delta_\Sigma + \nabla_d^2$ provided the extrinsic curvature of Σ vanishes and that

$$W_{acbc} = R_{acbc} + \delta_{ab}(\frac{1}{20}R - \frac{1}{4}R_{cc})\,. \tag{49}$$

Now, irrespectively of the presence or absence of the $O(2)$ symmetry we always have that

$$R_{acbc} = \frac{1}{2}\delta_{ab}R_{dcdc}\,. \tag{50}$$

Hence, the transformation (48) vanishes and the Wald entropy s_3 is indeed conformal invariant to linear order for a constant curvature geometry and a surface with vanishing extrinsic curvature. Notice, that for the HMS geometries we have that $R_{acbc} = 0$.

That the total Wald entropy (13) is not conformal invariant is thus due to the presence of the terms (31) which correspond to the total derivative terms in the conformal anomaly. Indeed, we find for the conformal transformation of these terms

$$\delta_\sigma S_W^{(C_4)} = -\delta_\sigma S_W^{(C_6)} = 16\pi \int_\Sigma [(R_{ab} - \frac{1}{2} R_{cc}\delta_{ab})\nabla_a \nabla_b \sigma] \,,$$
$$\delta_\sigma S_W^{(C_7)} = 0 \,, \tag{51}$$

where in the last line we used (45) and (49). Thus, to this order the entropy $S_W^{(C_7)}$ is conformal invariant while the non-invariance of the Wald entropy (13) is entirely due to the terms $S_W^{(C_4)}$ and $S_W^{(C_6)}$ (provided we use values (10) for the charges d_k).

7 The HMS discrepancy for a general 6d anomaly

The discrepancy discovered in [4] is a mismatch between the holographic calculation of the universal term in the entropy and the Wald entropy computed for the holographic conformal anomaly. However, in [4] it was considered only the BI-form of the conformal anomaly and the total derivative terms were neglected. As we explained in this note the actual holographic anomaly differs from the BI-form by certain total derivative terms. For the geometries considered in [4] the Wald entropy of these total derivatives is non-vanishing, as we have just demonstrated. Thus, the finding of [4] should be completed by adding the corresponding contributions of the total derivative terms.

For an arbitrary conformal anomaly taking the form (1) with the conformal charges b_k and the charges d_k the discrepancy takes the form

$$\begin{aligned}
\Delta S_{EE} &= 4\pi b_3 \ln\epsilon \int_\Sigma (W^{ab\mu\nu} W_{\mu\nu}{}^{ab} - W^{a\mu\nu\rho} W^a{}_{\mu\nu\rho} + 2W^{a\mu\nu a} W_\mu{}^{bb}{}_\nu - 2W^{b\mu\nu a} W_\mu{}^{ab}{}_\nu) \\
&-2\pi(d_4 - d_6)\ln\epsilon \int_\Sigma (R^2_{aa} - R^2_{ab} - R^2_{\mu a}) \\
&-2\pi d_7 \ln\epsilon \int_\Sigma [2\Box(R_{abab} - 2R_{aa} + R) + 3(R^2_{\mu a} - R_{\mu\nu}R^{\mu a\nu a}) \\
&+3W^{b\mu\nu a} W_\mu{}^{ab}{}_\nu - 3W^{a\mu\nu a} W_\mu{}^{bb}{}_\nu - \frac{3}{2} W^{ab\mu\nu} W_{\mu\nu}{}^{ab} + \frac{1}{2} W^{a\mu\nu\rho} W^a{}_{\mu\nu\rho} + 2R^a_\mu W^{\mu bab} - R_{\mu\nu} W^{\mu a\nu a}] \,,
\end{aligned} \tag{52}$$

where in the first line we used the result of [4]. This is a general form for the discrepancy. For the geometries more general than those considered in section 4.1 there may appear some terms with derivatives of the curvature in the first line of (52). The analysis of [4] does not allow us to identify those terms. We also notice that $d_4 - d_6 = b_2$ for the holographic anomaly obtained from the $d = 7$ Einstein action.

8 Conclusions

In this note we have analyzed the general form for the conformal anomaly in six dimensions and the corresponding Wald entropy. We have demonstrated that what the Wald entropy is concerned the total derivative terms which generically appear in the conformal anomaly can not be neglected. Their contribution is essential for the consistency when the entropy of the holographic conformal anomaly is compared to the entropy computed for the anomaly expressed in terms of the conformal invariants. The other important role of the total derivative terms is that their presence in the anomaly breaks the conformal invariance of the corresponding Wald entropy. Finally, we use our findings and give a general form for the discrepancy found in [4].

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